Homework 6 Solutions

Math 131B-2

- (9.2) (a) If $f(x) = x$, $|f(x) - f_n(x)| = \frac{|x|}{n}$ is bounded on any bounded interval. Ergo $f_n \to f$ uniformly. Moreover, if $g(x) = 0$ when x is irrational or 0 and $g(x) = b$ when $x = \frac{a}{b}$ is rational, and a and b are coprime, $b > 0$, then $|g(x) - g_n(x)| = \frac{1}{n}$. Ergo $g_n \to g$ uniformly.

  (b)Notice that $h_n(x)$ converges pointwise to $h_n(x) = 0$ when x is irrational or 0, and $h_n(x) = a$ when x is rational and can be written as $x = \frac{a}{b}$ such that a and b are coprime integers and $b > 0$. If h_n converges uniformly on a bounded interval $I \subset \mathbb{R}$, h_n must converge uniformly on some closed $[c, d]$ in I. Then given $\epsilon = 1$, there is some N such that $n \geq N$ implies that $|h(x) - h_n(x)| < \epsilon$ for all $x \in [c, d]$. Without loss of generality we can assume c and d have the same sign. Now observe that when $x = \frac{a}{b}$ is rational we have

$$\begin{aligned} |h(x) - h_n(x)| &= \left| \frac{a}{b}\left(1 + \frac{1}{n}\right)\left(b + \frac{1}{n}\right) - a \right| \\ &= \left| \frac{a}{n} + \frac{x}{n} + \frac{x}{n^2} \right| \\ &> \left| \frac{a}{n} \right|. \end{aligned}$$

  Here the last inequality follows from the observation that a and x have the same sign, since $b > 0$. However, we contend there are rational numbers $x = \frac{a}{b}$ with arbitrarily high a in any interval $[c, d]$. For suppose not. Then there would be some M such that if $x \in [c, d]$ and $x = \frac{a}{b}$, we have $|a| < M$. But $c < \frac{a}{b} < d$ implies that $bc < a < bd$. Therefore $M > |a| > \min\{|bd|, |bc|\}$. There are finitely many pairs a and b satisfying this equation for any c and d, but this is nonsense because we know there are infinitely many rationals in $[c, d]$. Therefore $|a|$ is not bounded on any interval $[c, d]$, implying that the convergence cannot $h_n \to h$ cannot be uniform.

- (9.3)(a) Straightforward triangle inequality. (b) Suppose that $f_n \to f$ and $g_n \to g$ are uniformly convergent, and f_n, g_n are all bounded. By (9.1) $\{f_n\}$ and $\{g_n\}$ are uniformly bounded, i.e. there exists M_1 such that $|f_n(x)| < M_1$ for all $n \in \mathbb{N}$ and $x \in S$ and M_2 such that $|g_n(x)| < M_2$ for $n \in \mathbb{N}$ and $x \in S$, and therefore $|g(x)| \leq M_2$ for $x \in X$. Let $M = \max\{M_1, M_2\}$. Given $\epsilon > 0$, choose N_1 such that $n \geq N_1$ implies $|f_n(x) - f(x)| < \frac{\epsilon}{2M}$ and N_2 such that $n \geq N_2$ implies $|g_n(x) - g(x)| < \frac{\epsilon}{2M}$. Then for

$n \geq N = \max\{N_1, N_2\}$ and any $x \in X$, we have the following for $h(x)$.

$$\begin{aligned} |h_n(x) - h(x)| &= |f_n(x)g_n(x) - f(x)g(x)| \\ &\leq |f_n(x)g_n(x) - f_n(x)g(x)| + |f_n(x)g(x) - f(x)g(x)| \\ &= |f_n(x)||g_n(x) - g(x)| + |f_n(x) - f(x)||g(x)| \\ &< M \cdot \frac{\epsilon}{2M} + \frac{\epsilon}{2M} \cdot M \\ &= \epsilon \end{aligned}$$

We conclude that $h_n \to h$ uniformly.

- (9.14) We see that the pointwise limit of $\{f_n\}$ is $f(x) = 0$ and the pointwise limit of $f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2}$ is the function $g(x)$ with the property that $g(0) = 1$ and $g(x) = 0$ for $x \neq 1$.

  (a) We see that $f'(x) = 0$ exists everywhere, but $f'(0) = 0 \neq 1 = g(0)$.

  (b) Note that $1 + nx^2 \geq 2\sqrt{n}|x|$, so for any $x \in \mathbb{R}$, we have $|f_n(x)| = \frac{|x|}{1+nx^2} \leq \frac{1}{2\sqrt{n}}$. We can use this bound, which does not depend on x, to show the convergence $f_n \to 0$ is uniform on $\mathbb{R}$.

  (c) Since each f_n' is continuous at 0 but $g(x)$ is not continuous at 0, our interval cannot contain 0. Let $[a, b]$ be any closed interval such that $a > 0$. Then we have

$$\begin{aligned} \left|\frac{1-nx^2}{(1+nx^2)^2}\right| &= \left|\frac{1-nx^2}{1+nx^2}\right| \cdot \frac{1}{1+nx^2} \\ &\leq \frac{1}{1+nx^2} \\ &\leq \frac{1}{na^2} \end{aligned}$$

  We can use this bound, which does not depend on x, to show $f_n \to 0$ uniformly on $[a, b]$. The situation is similar on $[a, b]$ when $b < 0$, only replacing a with b in the inequality above.

- (9.16) We know that $\int_0^1 f_n \to \int_0^1 f$ as a sequence of real numbers. Now, since $f_n \to f$ uniformly, by (9.1) $\{f_n\}$ is uniformly bounded, i.e. $|f_n(x)| < M$ for all $n \in \mathbb{N}$ and $x \in [0, 1]$. Ergo $|\int_{1-\frac{1}{n}}^1 f_n| < M(\frac{1}{n}) = \frac{M}{n}$. Therefore we see that $|\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f_n| \leq |\int_0^{1-\frac{1}{n}} f_n - \int_0^1 f_n| + |\int_0^1 f_n - \int_0^1 f| = |\int_{1-\frac{1}{n}}^1 f_n| + |\int_0^1 f_n - \int_0^1 f| < \frac{M}{n} + |\int_0^1 f_n - \int_0^1 f|$. The last term goes to zero as n goes to infinity, so in fact $\int_0^{1-\frac{1}{n}} f_n \to \int_0^1 f$.

- (9.22) We observe that $|a_n \sin(nx)| \leq |a_n|$ and likewise $|a_n \cos(nx)| \leq |a_n|$. Ergo by the Weierstrass M-test, if $\sum |a_n|$ converges the series $\sum_{i=1}^{\infty} a_n \sin(nx)$ and $\sum_{i=1}^{\infty} a_n \cos(nx)$ converge uniformly.

- *Dini's Theorem* Let $f_n : X \to \mathbb{R}$ be a sequence of functions on a compact metric space X which converges pointwise to a continuous function $f : X \to \mathbb{R}$ and suppose that for each x the sequence $\{f_n(x)\}$ is increasing, i.e. $f_n(x) \leq f_m(x)$ for all $n < m$. We will prove that $\{f_n\}$ in fact converges to f uniformly.

  - Let $g_n = f - f_n$. Then g_n is a continuous function, because f and f_n are continuous. Therefore $V_n^\epsilon = \{x \in X : |g_n(x)| < \epsilon\} = g_n^{-1}((-\infty, \epsilon))$ is the preimage of an open set under a continuous function, hence open. Moreover, since $n < m$ implies $f_n(x) \leq f_m(x)$ for all $x \in X$, we see that $n < m$ implies that $g_n(x) > g_m(x)$. Therefore $V_n^\epsilon \subseteq V_m^\epsilon$.

  - Because $f_n \to f$ pointwise, for each $x \in X$ there is some N_x such that $n \geq N_x$ implies that $0 \leq f - f_n(x) < \epsilon$. Ergo $x \in V_n^\epsilon$ for all $n \geq N_x$. Since this is true for all x, the sets $V_1^\epsilon \subset V_2^\epsilon \subset V_3^\epsilon \subset \cdots$ cover X. Ergo since X is compact, there is some finite subcover of the V_n^ϵ which cover X. But since the V_n^ϵ are ascending sets this just means there is some N such that V_N^ϵ contains X.

  - Since $X \subseteq V_n^\epsilon$ for some N, we know that $|g_n(x)| = |f - f_N(x)| < \epsilon$ for all $x \in X$. Moreover, since $n \geq N$ implies $f_n(x) \geq f_N(x)$, in fact whenever $n \geq N$, we see that $|f(x) - f_n(x)| < \epsilon$. Since ϵ was arbitrary, $f_n \to f$ uniformly.
  - *A question of arclength.* Let $f_n(x) = \frac{1}{n}\sin(nx)$, then $f_n'(x) = \cos(nx)$. Ergo the arclength $S_a^b(f_n) = \int_0^\pi \sqrt{1 + \cos^2(nx)}dx$. Observe that $\cos^2(nx) \geq \frac{1}{2}$ on a set A composed of the union of the intervals $[0, \frac{\pi}{4n}]$, $[\frac{(n-1)\pi}{4}, \pi]$, and $[\frac{(4k-1)\pi}{4n}, \frac{(4k+1)\pi}{4}]$ for all $1 \leq q \leq n$. The lengths of these intervals add up to $\frac{\pi}{2}$. Ergo on A, $\sqrt{1 + \cos^2(nx)} \geq \sqrt{\frac{3}{2}}$. On the remaining intervals $[\frac{(4\ell+1)\pi}{n}, \frac{(4\ell+3)\pi}{n}]$, whose lengths also sum to $\frac{\pi}{2}$ for $0 \leq \ell \leq n-1$, we have $\sqrt{1 + \cos^2(nx)} \geq 1$. Therefore we have the lower bound $\int_0^\pi \sqrt{1 + \cos^2(nx)}dx \geq \frac{\pi}{2}\sqrt{\frac{3}{2}} + \frac{\pi}{2}(1) > \pi$. Since the arclength of $f \equiv 0$ is π, we cannot have $S_a^b(f_n) \to S_a^b(f)$. We would have to add a hypothesis about uniform convergence of the derivatives to ensure convergent arclengths.

- Continuity makes life easier.

  - Since each f_n' is continuous on (a, b), their uniform limit g is continuous. Moreover, continuous functions on closed intervals are integrable, so $\int_{x_0}^x f_n'$ and $\int_{x_0}^x g$ exist for all $x \in (a, b)$ (bearing in mind that if $a < b$, the integral $\int_b^a f \equiv -\int_a^b f$). Ergo $\int_{x_0}^x f_n' \to \int_{x_0}^x g$. [There's one subtlety here: we proved uniform convergence of functions defined by the integral on a closed interval $[a, b]$ in class. But it remains the case that if $|f_n'(x) - g(x)| < \epsilon$ for all $x \in (a,b)$, $\int_c^d |f_n'(x) - g(x)| \leq \epsilon(b-a)$ for any closed interval $[c,d]$ in (a,b). So the proof extends with no issues.]

- Nothing to prove; $f_n(x) - f_n(x_0) \to \int_{x_0}^x g$ uniformly.

- Let $f : (a,b) \to \mathbb{R}$ defined by $f(x) = L + \int_{x_0}^x g$. Notice that $f_n(x) = f_n(x_0) + (f_n(x) - f_n(x_0))$. Now $f_n(x_0) \to L$ as a sequence of real numbers, so given $\epsilon > 0$ there is an N_1 such that $|f_n(x_0) - L| < \frac{\epsilon}{2}$. Moreover, by the second part of this problem, there is an N_2 such that $n > N_2$ implies that $|(f_n(x) - f_n(x_0)) - \int_{x_0}^x g| \leq \frac{\epsilon}{2}$ for all x in (a,b). Ergo, for $n \geq \max\{N_1, N_2\}$, we have $|fn(x) - f(x)| \leq |f_n(x) - L| + |(f_n(x) - f_n(x_0)) - \int_{x_0}^x g| < \epsilon$, and therefore $f_n \to f$ uniformly.

- Since g is continuous, by FTC Part II, $f'(x) = 0 + g(x)$.

Citation: This outline is based on the proof in Tao's *Analysis II*.